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Distribution of zeros of the partition function of the antiferromagnetic Husimi–Temperley model I

M OHMINAMI†, Y ABE and S KATSURA
Department of Applied Physics, Tohoku University, Sendai, Japan

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Abstract. To study the locus of zeros of the partition function for the antiferromagnets, an exactly soluble simplest model (the antiferromagnetic Husimi–Temperley model) is proposed. Thermodynamic properties of the model are shown to be the same as those in the molecular field approximation of the Ising or Heisenberg model. The locus of zeros in the complex fugacity plane of the model is obtained by the method of comparing the real part of the various branches of the complex free energies (in most cases, of the antiferromagnetic state and of the paramagnetic states). The locus is a loop which does not cut the real positive axis above the critical temperature. Below the critical temperature, the locus of the zeros consists of dual loops which cut the real positive axis at z_c and $1/z_c$ and the density of zeros near the real axis is proportional to $|z-z_c|$. At the critical temperature the locus is a loop which crosses at $z = 1$ and the density of zeros is proportional to $|z-z_c|^3$.

1. Introduction

Yang and Lee (1952) related phase transitions to the distribution of zeros of the partition function in the complex fugacity plane, and Lee and Yang (1952) proved the circle theorem for the ferromagnetic Ising model of $S = \frac{1}{2}$. Much work has been published since, and in particular, it was proved that the circle theorem holds for several ferromagnetic systems (Griffiths 1969, Asano 1970, Suzuki and Fisher 1971).

For the antiferromagnetic systems, the situation is more complicated. Zeros of the partition function of the one dimensional antiferromagnetic Ising model were proved to be located on the negative real axis (Yang 1952). Katsura *et al* (1971, to be referred to as KAY) examined the finite (4×6) Ising model with nearest (J) and next nearest (J') neighbour interactions and found several patterns of the distribution of zeros. In the case $J < 0$ and $J' > 0$, where the antiferromagnetic configuration is more stable than the case $J' = 0$, the locus at low temperatures is found to consist of two nearly concentric circles which cut the positive real axis at z_c and $1/z_c$ $(= \exp(\pm 2mH_c/kT))$ corresponding to the critical field. Katsura and Ohminami (1972, to be referred to as KO) obtained the distribution of zeros of the one dimensional Ising model with nearest and next nearest neighbour interactions by the transfer matrix method and showed that the locus consists of a part of two nearly concentric circles for the case $J < 0$ and $J' > 0$.

In this paper, a soluble model for the antiferromagnet—the antiferromagnetic Husimi–Temperley (AHT) model, speaking more generally, two 'sublattice' Husimi–

† Now at The Electrical Communication Laboratory, Musashino, Tokyo, Japan.

Temperley model—is introduced in § 2 and thermodynamic properties of the model are shown to be the same as the results in the molecular field approximation of the Ising or the Heisenberg model (Van Vleck 1941, Garrett 1952). By a method described in § 3 the locus of zeros for the AHT model is obtained. Above the critical temperature the locus is a loop which does not cut the positive real axis. Below the critical temperature the locus consists of dual loops which cut the positive real axis at z_c and $1/z_c$. The density of the zeros of this model is discussed in § 4.

2. Two sublattice (antiferromagnetic) Husimi–Temperley model

We consider a system of Ising spins of $S = \frac{1}{2}$ which are located on either an α 'sublattice' or β 'sublattice'. The number of sites of each sublattice is $N/2$. The number of down spins (opposite to the magnetic field) on the α (β) sublattice is denoted by n_α (n_β). All interactions between sites in the same or in the different sublattices are assumed to be equal. The pair interaction energy is taken to be $-J/N$ ($+J/N$) between spins of the same (opposite) directions on different sublattices, and $-\gamma J/N$ ($+\gamma J/N$) between spins of the same (opposite) directions on the same sublattices.

The partition function $Z(T, H, N)$ of the model is given by

$$Z(T,H,N) = z^{-N/2} \sum_{n_\alpha=0}^{N/2} \sum_{n_\beta=0}^{N/2} \binom{N/2}{n_\alpha}\binom{N/2}{n_\beta} z^{n_\alpha+n_\beta}$$

$$\times \exp\left[\frac{J}{NkT}\left(\frac{N}{2}-2n_\alpha\right)\left(\frac{N}{2}-2n_\beta\right) + \frac{\gamma J}{2NkT}\left\{\left(\frac{N}{2}-2n_\alpha\right)^2+\left(\frac{N}{2}-2n_\beta\right)^2-N\right\}\right] \tag{2.1}$$

where $z = \exp(-2mH/kT)$ and m is the magnetic moment of the spin.

When $\gamma = 1$, the difference between the α and β sublattices disappears, and the exponent becomes a function of only $n_\alpha + n_\beta \equiv n$. Carrying out one of the summations, we have:

$$Z(T,H,N) = z^{-N/2} \sum_{n=0}^{N} \binom{N}{n} z^n \exp\left(\frac{J}{2NkT}\{(N-2n)^2-N\}\right). \tag{2.2}$$

Equation (2.2) represents the partition function of the (one sublattice) Husimi–Temperley model, for which a molecular field approximation of the ferromagnets was shown to be valid exactly when $J > 0$ (Husimi 1953, Temperley 1954, Katsura 1955). Zeros of the partition function (2.2) for $J > 0$ were found to lie on the unit circle in the z plane and the distribution function was given (Katsura 1955, and references in KAY).

When $J < 0$ and $\gamma = 1$, the (one sublattice) Husimi–Temperley model with antiferromagnetic interaction comes out. The zeros of the partition function of this model were proved to lie on the negative real axis by Heilmann (1971), and the model does not show the phase transition.

The model introduced here (equation (2.1)) is the Husimi–Temperley model generalized in such a way that the antiferromagnetism can be treated. When $J > 0$ and $\gamma \geqslant 0$, zeros distribute on the unit circle and the property is ferromagnetic.

Now the case of $J < 0$ and $\gamma \leqslant 0$, that is the antiferromagnetic case, is considered. The free energy F_N of this system is given by

$$\frac{F_N}{NkT} = \frac{1}{2t}\{\sigma_\alpha\sigma_\beta + \tfrac{1}{2}\gamma(\sigma_\alpha^2+\sigma_\beta^2)\} - \frac{h}{2t}(\sigma_\alpha+\sigma_\beta) + \tfrac{1}{4}\{(1-\sigma_\alpha)\ln(1-\sigma_\alpha) + (1+\sigma_\alpha)\ln(1+\sigma_\alpha)$$
$$+(1-\sigma_\beta)\ln(1-\sigma_\beta)+(1+\sigma_\beta)\ln(1+\sigma_\beta)\} - \ln 2, \tag{2.3}$$

$$t = 2kT/(-J), \qquad h = 2mH/(-J), \tag{2.4}$$

where t, h, σ_α ($= 1-4n_\alpha/N$) and σ_β ($= 1-4n_\beta/N$) denote the temperature, the magnetic field, α and β sublattice magnetizations in reduced units. Sublattice magnetizations σ_α and σ_β are determined by minimizing the free energy:

$$\sigma_\alpha = \tanh\left(\frac{h-\sigma_\beta-\gamma\sigma_\alpha}{t}\right) \tag{2.5a}$$

$$\sigma_\beta = \tanh\left(\frac{h-\sigma_\alpha-\gamma\sigma_\beta}{t}\right). \tag{2.5b}$$

Equations (2.5a) and (2.5b) have two solutions. The paramagnetic state is given by $\sigma_\alpha = \sigma_\beta = \sigma_p$, that is,

$$\sigma_p = \tanh\left(\frac{h-(1+\gamma)\sigma_p}{t}\right). \tag{2.5c}$$

The antiferromagnetic state is characterized by $\sigma_\alpha \neq \sigma_\beta$. The normalized total magnetization $\sigma_a = (\sigma_\alpha+\sigma_\beta)/2$ is given by

$$\frac{1}{\sigma_a}\sinh\left(\frac{2h-2(1+\gamma)\sigma_a}{t}\right) = 2\frac{1-\gamma}{t}$$
$$\times\left[\left\{\frac{1}{\sigma_a}\sinh\left(\frac{2h-2(1+\gamma)\sigma_a}{t}\right) - \cosh\left(\frac{2h-2(1+\gamma)\sigma_a}{t}\right)\right\}^2 - 1\right]^{1/2}$$
$$\times\left[\operatorname{arccosh}\left\{\frac{1}{\sigma_a}\sinh\left(\frac{2h-2(1+\gamma)\sigma_a}{t}\right) - \cosh\left(\frac{2h-2(1+\gamma)\sigma_a}{t}\right)\right\}\right]^{-1}, \tag{2.6}$$

as a function of the magnetic field. Putting $\sigma_a = \sigma_p = \sigma_0$ in equations (2.5c) and (2.6), the phase boundary between the paramagnetic phase and the antiferromagnetic phase is found to be

$$h_c = t\operatorname{arctanh}\sigma_0 + (1+\gamma)\sigma_0 \tag{2.7a}$$

$$\sigma_0 = \pm\left(1-\frac{t}{t_c}\right)^{1/2} \tag{2.7b}$$

$$t_c = 1-\gamma, \tag{2.8}$$

where t_c is the reduced Néel temperature.

These results mean that the thermodynamic properties of our model are the same as those of the molecular field approximation (Van Vleck 1941, Garrett 1952) of the antiferromagnets. We call the model the 'two sublattice' Husimi–Temperley model, and in particular when $J < 0$ and $\gamma \leqslant 0$, the antiferromagnetic Husimi–Temperley (AHT) model. It is expected that the distribution of zeros of the AHT model represents the behaviour of that of the antiferromagnetic Ising or Heisenberg model qualitatively well.

3. The locus and the density of zeros

We assume that the zeros of the partition function distribute on a curve C in the complex z plane in the limit $N \to \infty$. The number of zeros in a line element $\mathrm{d}s$ at $z = z(s)$ is denoted by $Ng(s)\,\mathrm{d}s$. Then χ is defined by

$$\chi(z, t) + 2\ln 2 \equiv \lim \frac{2}{N}\ln Z_N = -\lim \frac{2F_N}{NkT} = \int_{\mathrm{C}} g(s)\ln\left(1 - \frac{z}{z(s)}\right)\mathrm{d}s, \quad (3.1)$$

$$z\chi'(z) = \sigma = \int_{\mathrm{C}} \frac{zg(s)\,\mathrm{d}s}{z - z(s)}, \quad (3.2)$$

where $z = \exp(-2h/t)$ is the fugacity.

First we consider the case where the partition function Z_N is expressed in terms of the eigenvalues λ_i of the transfer matrix (KO). We denote as λ_1 a branch which has the largest absolute value among λ_i, then we have

$$Z_N = \sum \lambda_i^N \sim \lambda_1^N. \quad (3.3)$$

When the absolute values of λ_1 and λ_2 are equal ($\lambda_2 = \lambda_1\,\mathrm{e}^{\mathrm{i}\psi}$), we have

$$Z_N \sim \lambda_1^N(1 + \mathrm{e}^{\mathrm{i}N\psi}). \quad (3.4)$$

The right hand side of equation (3.3) cannot be zero while that of (3.4) can. Thus we see that when $Z_N(z) = 0$, $|\lambda_1(z)|$ should be equal to $|\lambda_2(z)|$. In most cases $\lambda_2(z)$ is a branch generated by an analytic continuation of $\lambda_1(z)$, and $|\lambda_1(z)|$ is equal to $\exp\{-\mathrm{Re}(F_N/NkT)\}$, where F_N is a complex free energy continued analytically from the real axis.

From the above argument we assume that the locus C of zeros is generally obtained in the following way. At a given temperature t, two complex solutions of equations (2.5a) and (2.5b), $\sigma_1(z)$ and $\sigma_2(z)$, which make the real part of $\chi(z)$ the largest and the second largest, are calculated. The set of z which satisfies

$$\mathrm{Re}\,\chi(\sigma_1(z)) = \mathrm{Re}\,\chi(\sigma_2(z)), \quad (3.5)$$

where

$$\chi(z) = \chi(\sigma_{\mathrm{a}}(z)) = \frac{1}{t}\sigma_\alpha\sigma_\beta + \frac{\gamma}{2t}(\sigma_\alpha^2 + \sigma_\beta^2) - \frac{1}{2}\ln(1 - \sigma_\alpha^2) - \frac{1}{2}\ln(1 - \sigma_\beta^2), \quad (3.6a)$$

or

$$\chi(z) = \chi(\sigma_{\mathrm{p}}(z)) = \frac{1}{t}(1 + \gamma)\sigma_{\mathrm{p}}^2 - \ln(1 - \sigma_{\mathrm{p}}^2), \quad (3.6b)$$

is regarded as a locus of zeros. Then the density $g(z)$ is given by

$$g(z) = \frac{1}{2\pi}\frac{1}{|z|}|\sigma_1 - \sigma_2|. \quad (3.7)$$

In most cases σ_1 and σ_2 are the magnetizations of the complex paramagnetic phase and the complex antiferromagnetic phase (see § 4). In some cases they are different branches of a paramagnetic phase (see Appendix). The circle theorem in the ferromagnetic case is also shown to be consistent with the above procedure (see Appendix).

The loci of zeros thus obtained are shown in figure 1 for $\gamma = 0$ and figure 2 for $\gamma = -1$. The reduced temperature t/t_c is taken to be, (*a*) below, (*b*) at, and (*c*) above the Néel temperature in figures 1 and 2. In figure 1(*b*), zeros of finite systems of $N = 60$, and in figure 2(*a*), those of $N = 60$, 16 and 8, are also plotted. It is confirmed that the zeros of a finite model tend to lie on the locus obtained by the present method† as N goes to infinity. It is to be noted that the locus of zeros in the antiferromagnetic case is of an asymptotic nature, that is, zeros tend to distribute on the locus obtained here as N goes to infinity, while in the ferromagnetic case the circle theorem holds for any finite N.

The locus consists of double loops below $t < t_c$, of one loop above $t > t_c$, and of a hybrid shape at $t = t_c$. The points at which the loops cut the positive real axis correspond to the critical fields given by equation (2.8). When $\gamma = 0$, the loops have cusps on the real axis and tails appear on the negative real axis, while when $\gamma = -1$, tails disappear. The loop part of the locus was obtained as the place where the real part of the complex free energy of the complex antiferromagnetic state and that of the complex paramagnetic state become equal, while the tail part was where those of the two complex paramagnetic states become equal (see Appendix). Such patterns of the locus of zeros of the antiferromagnets are what KAY predicted.

The discussion of the distribution of zeros near the real axis and its relation to the critical thermodynamic properties are given in § 4.

4. Distribution of zeros near the positive real axis

In this section, the distribution of zeros near the positive real axis for the case $\gamma = 0$ is studied. From equations (2.5*a*) and (2.5*b*), the fugacity z is related to:

$$z = \frac{1-\sigma_\alpha}{1+\sigma_\alpha}\exp\left(\frac{-2\sigma_\beta}{t}\right), \tag{4.1a}$$

or to an equation in which α and β are interchanged in (4.1*a*) which is denoted by (4.1*b*). Such conventions are used hereafter.

4.1. *Below the critical temperature* ($t < 1$)

The sublattice magnetization of the AHT model at the critical field is given by equation (2.7). Put

$$\hat{h} = h - h_c \tag{4.2}$$

$$\hat{z} \equiv z/z_c - 1 = \exp(-2\hat{h}/t) - 1 = r\,e^{i\psi} \tag{4.3}$$

$$\hat{\sigma}_\alpha \equiv \sigma_\alpha - \sigma_0, \qquad \hat{\sigma}_\beta \equiv \sigma_\beta - \sigma_0, \tag{4.4}$$

where

$$z_c \equiv \exp(-2h_c/t) = \frac{1-\sigma_0}{1+\sigma_0}\exp\left(\frac{-2\sigma_0}{t}\right). \tag{4.5}$$

† In the case of a one dimensional hard core system, the procedure is proved to be rigorous by Penrose and Elvey (1968).

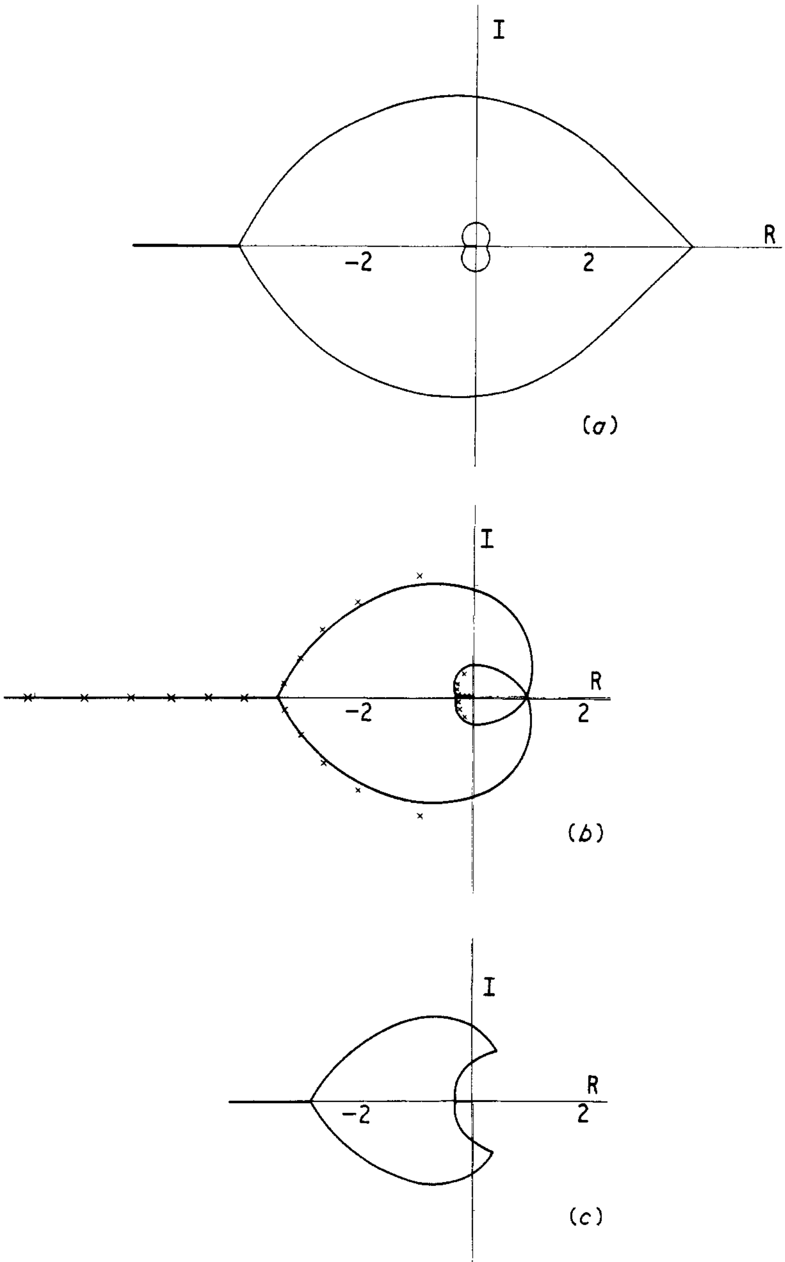

Figure 1. The locus of the zeros of the partition function for the case $\gamma = 0$. (*a*) $t/t_c = 0{\cdot}9$; (*b*) $t/t_c = 1{\cdot}0$; (*c*) $t/t_c = 1{\cdot}1$. In (*b*) crosses represent zeros for the case $N = 60$ (some zeros on the negative real axis are outside the figure).

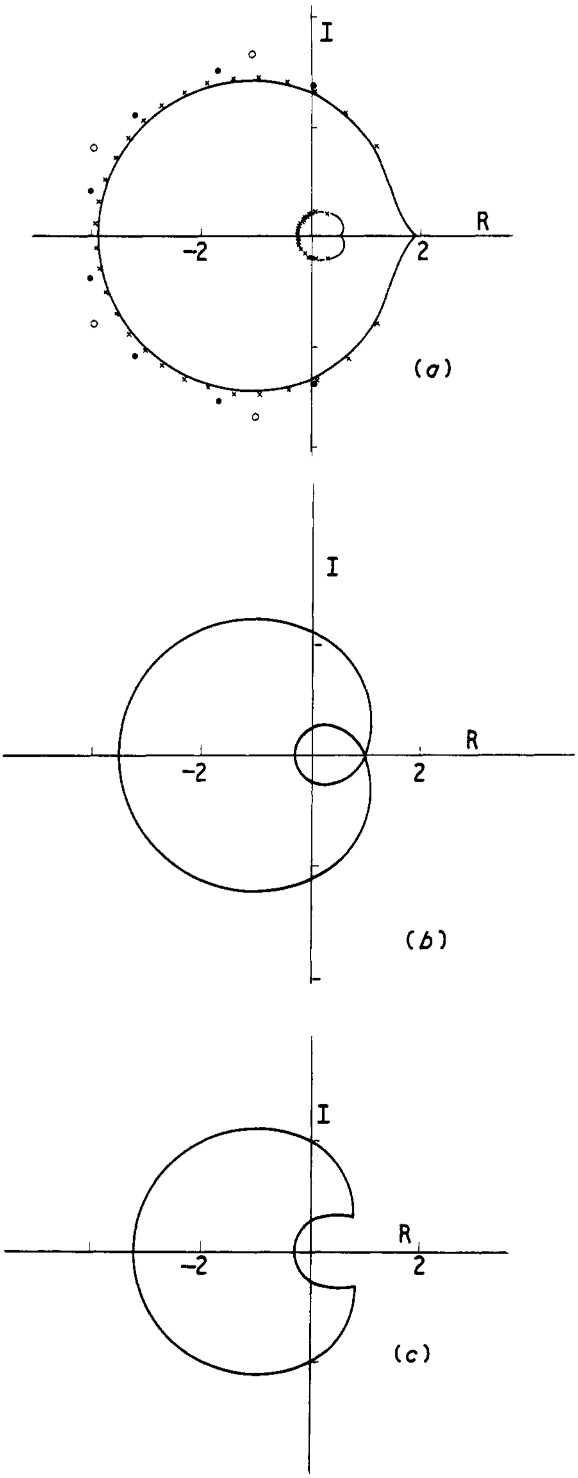

Figure 2. The locus of the zeros of the partition function for the case $\gamma = -1$. (*a*) $t/t_c = 0.9$; (*b*) $t/t_c = 1.0$; (*c*) $t/t_c = 1.1$. In (*a*) ×, ● and ○ represent zeros for the cases $N = 60$, 16 and 8 respectively (○ and ● in the inner loop are omitted).

Expanding the equations (3.6), (4.1*a*) and (4.1*b*) in terms of small quantities $\hat{\sigma}_\alpha$ and $\hat{\sigma}_\beta$, we have

$$\hat{\chi} \equiv \chi - \chi_c = \chi - \chi(z_c) = -2\sigma_0 t^{-1}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta) + \tfrac{1}{2}t^{-1}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta)^2 - \sigma_0^2 t^{-2}(\hat{\sigma}_\alpha^2 + \hat{\sigma}_\beta^2) + \ldots \tag{4.6}$$

$$\hat{z} = -2t^{-1}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta) + 2t^{-2}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta)^2 - \tfrac{4}{3}t^{-3}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta)^3$$
$$-2\sigma_0 t^{-2}\hat{\sigma}_\alpha^2 - \tfrac{2}{3}t^{-3}\hat{\sigma}_\alpha^3 + 2\hat{\sigma}_0 t^{-3}\hat{\sigma}_\alpha^2(2\sigma_0 - \sigma_0\hat{\sigma}_\alpha + 2\hat{\sigma}_\beta) + \ldots. \tag{4.7a}$$

From equation (4.7), we have

$$\hat{z} = -2t^{-1}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta) + 2t^{-2}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta)^2 - \sigma_0 t^{-2}(\hat{\sigma}_\alpha^2 + \hat{\sigma}_\beta^2) + \ldots \tag{4.8}$$

and

$$0 = (\hat{\sigma}_\alpha - \hat{\sigma}_\beta)\{-2\sigma_0 t^{-2}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta) - 2t^{-3}(\tfrac{1}{3} - 2\sigma_0 + \sigma_0^2)(\hat{\sigma}_\alpha^2 + \hat{\sigma}_\alpha\hat{\sigma}_\beta + \hat{\sigma}_\beta^2) + 4\sigma_0 t^{-3}\hat{\sigma}_\alpha\hat{\sigma}_\beta\} + \ldots. \tag{4.9}$$

From equation (4.9), two solutions are obtained.

4.1.1. The p-state. A state where $\hat{\sigma}_\alpha = \hat{\sigma}_\beta$ is satisfied, is called a complex paramagnetic state. Then

$$\hat{z} = -2t^{-1}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta) + \ldots \tag{4.10}$$

$$\hat{\chi}_p = \hat{z}\sigma_0 - \frac{t}{8}\left(1 + \frac{4\sigma_0}{t}\right)\hat{z}^2 + O(\hat{z}^3). \tag{4.11}$$

4.1.2. The a-state. A state where equation (4.9) is satisfied with $\sigma_\alpha \neq \sigma_\beta$ is called a complex antiferromagnetic state. Then

$$\hat{z} = -2t^{-1}(\hat{\sigma}_\alpha + \hat{\sigma}_\beta)(1 - \sigma_0^2\rho^{-1}) + \ldots \tag{4.12}$$

$$\hat{\chi}_a = \hat{z}\sigma_0 - \tfrac{1}{8}t(1 + 4\sigma_0 t^{-1})(1 - \sigma_0^2\rho^{-1})^{-2}\hat{z}^2 + O(\hat{z}^3), \tag{4.13}$$

where

$$\rho \equiv \tfrac{1}{3} + \sigma_0^2.$$

The difference between the χ of the p- and a-states is given by

$$\hat{\chi}_a - \hat{\chi}_p = -\frac{t\sigma_0}{8}\hat{z}^2(1 + 4\sigma_0 t^{-1})(\sigma_0 + 6). \tag{4.14}$$

Hence, $\cos 2\psi$ should be zero on the locus of zeros near z_c. From the condition of stability of the solutions we have $\sigma_0 \cos\psi > 0$, hence we have $z - z_c = r\exp(\pm\frac{3}{4}\pi i)$ for $z_c > 1$ and $z - z_c = r\exp(\pm\frac{1}{4}\pi i)$ for $z_c < 1$. In figure 2(*a*) ψ is observed to be $\pm\frac{1}{4}\pi$ and $\pm\frac{3}{4}\pi$. The density of zeros is given by

$$g(r) = \frac{t\sigma_0}{4\sqrt{2}}(1 + 4\sigma_0 t^{-1})(\sigma_0 + 6)r + \ldots. \tag{4.15}$$

It is linear to the distance from z_c, which implies $|\sigma - \sigma_0| = \text{constant} \times |h - h_c|$.

4.2. At the critical temperature ($t = 1$)

Putting

$$\hat{z} = z - z_c = z - 1 \tag{4.16}$$

in equations (4.1*a*) and (4.1*b*) and expanding equations (3.6), (4.1*a*) and (4.1*b*) in terms of small quantities σ_α and σ_β, we find

$$\chi = -\tfrac{1}{2}(\sigma_\alpha+\sigma_\beta)^2-\tfrac{1}{4}(\sigma_\alpha^4+\sigma_\beta^4)+\ldots \tag{4.17}$$

$$\hat{z} = -2(\sigma_\alpha+\sigma_\beta)+2(\sigma_\alpha+\sigma_\beta)^2-\tfrac{4}{3}(\sigma_\alpha+\sigma_\beta)^3-\tfrac{2}{3}\sigma_\alpha^3+\ldots . \tag{4.18}$$

From equation (4.18), we have

$$\hat{z} = -2(\sigma_\alpha+\sigma_\beta)+2(\sigma_\alpha+\sigma_\beta)^2-\tfrac{5}{3}(\sigma_\alpha+\sigma_\beta)^3+\sigma_\alpha\sigma_\beta(\sigma_\alpha+\sigma_\beta)+\ldots \tag{4.19}$$

and

$$0 = \sigma_\alpha^3-\sigma_\beta^3 = (\sigma_\alpha-\sigma_\beta)(\sigma_\alpha^2+\sigma_\alpha\sigma_\beta+\sigma_\beta^2)+\ldots . \tag{4.20}$$

The two states corresponding to the two solutions of equation (4.20) are the p-state and the a-state, respectively.

4.2.1. The p-state. Putting $\sigma_\alpha = \sigma_\beta = u$ in equations (4.17) and (4.19), we have

$$\chi_{\rm p} = -2u^2-\tfrac{1}{2}u^4+{\rm O}(u^6) \tag{4.21}$$

$$\hat{z} = -4u+8u^2-\tfrac{34}{3}u^3+{\rm O}(u^4). \tag{4.22}$$

Elimination of u between (4.21) and (4.22) yields

$$\chi_{\rm p} = -\tfrac{1}{8}\hat{z}^2+\tfrac{1}{8}\hat{z}^3-\tfrac{175}{1536}\hat{z}^4+{\rm O}(\hat{z}^5). \tag{4.23}$$

4.2.2. The a-state. Similarly, putting $(\sigma_\alpha+\sigma_\beta)^2 = \sigma_\alpha\sigma_\beta = 4u^2$, we have

$$\chi_{\rm a} = -2u^2+4u^4+{\rm O}(u^6) \tag{4.24}$$

$$\hat{z} = -4u+8u^2-\tfrac{8}{3}u^3+{\rm O}(u^4). \tag{4.25}$$

Then

$$\chi_{\rm a} = -\tfrac{1}{8}\hat{z}^2+\tfrac{1}{8}\hat{z}^3-\tfrac{23}{192}\hat{z}^4+{\rm O}(\hat{z}^5). \tag{4.26}$$

The difference of $\chi_{\rm a}$ and $\chi_{\rm p}$ is given by

$$\chi_{\rm a}-\chi_{\rm p} = -\frac{3}{2^9}\hat{z}^4+{\rm O}(\hat{z}^5). \tag{4.27}$$

Hence $\cos 4\psi$ should be zero on the locus. From the condition of stability of the solutions, we have $\cos 2\psi \leqslant 0$. Then the loci of zeros near $z = z_{\rm c}$ are

$$z = 1 \pm r\exp({\rm i}\tfrac{3}{8}\pi) \qquad \text{and} \qquad z = 1 \pm r\exp({\rm i}\tfrac{5}{8}\pi). \tag{4.28}$$

The result of (4.28) is observed in figure 2(*b*). From equation (4.27), the density of zeros near the real axis is given by

$$g(r) = \frac{1}{2\pi}\left|\frac{\partial}{\partial r}\{{\rm Im}(\chi_{\rm a}-\chi_{\rm p})\}\right| = \frac{3\sin(3\pi/8)}{2^8\pi}r^3. \tag{4.29}$$

Equation (4.29) implies that $|\sigma| = \text{constant}\times|h|^3$ at $t = t_{\rm c}$.

5. Conclusions

In the present paper an exactly soluble simplest model for the antiferromagnet (the AHT model) has been introduced and the distribution of zeros of the partition function of the model has been obtained. It has been shown that the thermodynamic properties of the AHT model are the same as those in the molecular field approximation for the Ising or the Heisenberg model.

The locus of zeros in the complex fugacity plane was obtained by the method of comparing the real part of the complex free energies of various branches of the states (the antiferromagnetic state and the paramagnetic state, or two branches of the paramagnetic state). That is, the line where the largest and the second largest of Re $\chi(z)$ become equal is the locus. The present consideration throws light on the method of Hemmer and Hauge (1964) and Hemmer *et al* (1966). The validity of the method was confirmed by comparing with the results of finite systems. The method of the present paper can be applied to other models, and will be the subject of other papers.

Above the critical temperature the locus is a closed loop which does not cut the positive real axis. Below the critical temperature the locus consists of dual closed loops enclosing the origin. The locus cuts the positive real z axis at z_c and $1/z_c$ corresponding to the critical fields.

It was also shown that below the critical temperature the locus for $\gamma = 0$ crosses the positive real z axis at angles $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$ and that the density near z_c is proportional to the distance from z_c, which means that $|\sigma_a - \sigma_0| = \text{constant} \times |h - h_c|$. At the critical temperature the locus crosses the positive real z axis at angles $\frac{3}{8}\pi$ and $\frac{5}{8}\pi$ and the density near z_c is proportional to the third power of the distance from z_c, which means that $|\sigma_a| = \text{constant} \times |h|^3$.

The results of the present paper clarify the characteristic features of the distribution of zeros of antiferromagnets as predicted in KAY and KO.

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Appendix. The unit circle and real axis as possible loci

Relationships between the real and imaginary parts of χ_a, χ_p, σ_α, σ_β, σ_p and h are derived from equations (2.5), (2.6) and (3.6). The case $\gamma = 0$ is considered. Put

$$h = h_R + ih_I \tag{A.1}$$

$$\sigma_\alpha = x_\alpha + iy_\alpha, \qquad \sigma_\beta = x_\beta + iy_\beta \tag{A.2}$$

$$\sigma_p = x_p + iy_p, \tag{A.3}$$

then we have:

$$h_R = x_\beta + \frac{t}{2} \operatorname{arctanh} \frac{2x_\alpha}{1 + x_\alpha^2 - y_\alpha^2} \tag{A.4a}$$

$$h_I = y_\beta + \frac{t}{2} \arctan \frac{2y_\alpha}{1 - x_\alpha^2 - y_\alpha^2}. \tag{A.5a}$$

Equations derived by the interchange of α and β in equations (A.4*a*) and (A.5*a*) hold and are denoted by equations (A.4*b*) and (A.5*b*).

$$h_{\mathrm{R}} = x_{\mathrm{p}} + \frac{t}{2} \operatorname{arctanh} \frac{2x_{\mathrm{p}}}{1 + x_{\mathrm{p}}^2 - y_{\mathrm{p}}^2} \tag{A.6}$$

$$h_{\mathrm{I}} = y_{\mathrm{p}} + \frac{t}{2} \arctan \frac{2y_{\mathrm{p}}}{1 - x_{\mathrm{p}}^2 - y_{\mathrm{p}}^2}, \tag{A.7}$$

and

$$\chi(\sigma_{\mathrm{a}}) = [\tfrac{1}{2}(x_\alpha x_\beta - y_\alpha y_\beta) - \tfrac{1}{4}\ln\{(1 - x_\alpha^2 + y_\alpha^2)^2 + 4x_\alpha^2 y_\alpha^2\} - \tfrac{1}{4}\ln\{(1 - x_\beta^2 + y_\beta^2)^2 + 4x_\beta^2 y_\beta^2\}] + \mathrm{i}\left\{\frac{1}{2}\arctan\left(\frac{2x_\alpha y_\alpha}{1 - x_\alpha^2 + y_\alpha^2}\right) - \frac{1}{2}\arctan\left(\frac{2x_\beta y_\beta}{1 - x_\beta^2 + y_\beta^2}\right)\right\} \tag{A.8}$$

$$\chi(\sigma_{\mathrm{p}}) = [\tfrac{1}{2}(x_{\mathrm{p}}^2 - y_{\mathrm{p}}^2) - \tfrac{1}{2}\ln\{(1 - x_{\mathrm{p}}^2 + y_{\mathrm{p}}^2)^2 + 4x_{\mathrm{p}}^2 y_{\mathrm{p}}^2\}] + \mathrm{i}\arctan\left(\frac{2x_{\mathrm{p}} y_{\mathrm{p}}}{1 - x_{\mathrm{p}}^2 + y_{\mathrm{p}}^2}\right). \tag{A.9}$$

When a set of $\sigma_\alpha = x_\alpha + \mathrm{i}y_\alpha$ and $\sigma_\beta = x_\beta + \mathrm{i}y_\beta$ is a solution of equation (A.4) for $h_{\mathrm{R}} = 0$, then a set of $\sigma_\alpha = -x_\alpha + \mathrm{i}y_\alpha$ and $\sigma_\beta = -x_\beta + \mathrm{i}y_\beta$ is also a solution of equation (A.4) which gives the same value of the real part of equation (A.8). It means that on the unit circle of the z plane Re $\chi(z)$ of these two branches of solutions are equal. When a set of $\sigma_\alpha = x_\alpha + \mathrm{i}y_\alpha$ and $\sigma_\beta = x_\beta + \mathrm{i}y_\beta$ is a solution of equation (A.4) for $h_{\mathrm{I}} = 0$, then a set of $\sigma_\alpha = x_\alpha - \mathrm{i}y_\alpha$ and $\sigma_\beta = x_\beta - \mathrm{i}y_\beta$ is also a solution of equation (A.4) which gives the same value of the real part of equation (A.8). It means that on the real axis of the z plane Re $\chi(z)$ of these branches of solutions are equal. What part of a unit circle or of a real axis can be a locus, depending on whether Re $\chi(z)$ on it is the largest (among various branches) or not? In the case $\gamma = 0$, tails which are parts of the negative real axis appear as an example of the latter.

When $\sigma_{\mathrm{a}} = (\sigma_\alpha + \sigma_\beta)/2 = \sigma_{\mathrm{p}}$ at some value of complex h, it is evident that Re $\chi(\sigma_{\mathrm{a}})$ = Re $\chi(\sigma_{\mathrm{p}})$. Unfortunately $\sigma_{\mathrm{a}} = \sigma_{\mathrm{p}}$ can occur only for real values of h (given by equation (2.8)). A numerical search to find the position of h where Re $\chi(\sigma_{\mathrm{a}})$ = Re $\chi(\sigma_{\mathrm{p}})$ with equations (A.4), (A.5), (A.8), and (A.6), (A.7), (A.9) gives loops in figures 1 and 2.

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